Parallel Branch and Bound Algorithm for Computing Maximal Structured Singular Value *

Xinjia Chen and Kemin Zhou
Department of Electrical and Computer Engineering
Louisiana State University
Baton Rouge, LA 70803
chan@ece.lsu.edu kemin@ece.lsu.edu

January 1, 2002, Revised on March 18, 2003

Abstract

In this paper, we have developed a parallel branch and bound algorithm which computes the maximal structured singular value μ without tightly bounding μ for each frequency and thus significantly reduce the computational complexity.

Keywords: Robust control, structured singular value, branch and bound.

1 Introduction

It is well known that the analysis of robust stability and performance with structured uncertainty boils down to the problem of computing the supremum of the structured singular value over all frequency [3, 11]. That is, $\mu_{max} := \sup_{\omega\in\mathbf{R}} \mu_{\mathbf{\Delta}}(M(j\omega))$ where $M(s)$ is the transfer function of the generalized system and $\mathbf{\Delta}$ is a set of block structured uncertainties. Related to this problem are the method proposed by Lawrence, Tits and Dooren [7, 8] and the approach established by Ferreres and Biannic [5, 6]. These interesting techniques can be applied to compute a μ upper bound over a frequency interval without gridding of frequency. For the precise computation of the maximal structured singular value μ_{max} (i.e., a tight lower bound is also expected in addition to an upper bound), the conventional method is to grid a range of frequency and compute the maximal μ among all the frequencies [1]. Since the exact computation is in general impossible, μ is obtained for each frequency by tightly bounding. Sophisticated upper bounds and lower bounds have been derived for

*This research was supported in part by grants from AFOSR (F49620-94-1-0415), ARO (DAAH04-96-1-0193), and LEQSF (DOD/LEQSF(1996-99)-04).

example in [2, 3, 4, 10] and techniques such as branch and bound [9] have been developed to refine the bounds.

It is noted that the existing techniques for computing the maximal structured singular value μ_{max} lack of efficiency because of the tedious frequency sweeping. In this paper, we investigate a smart frequency sweeping strategy. More specifically, we apply branch and bound scheme to compute μ for $N > 1$ frequencies in parallel. We introduce a powerful "pruning " mechanism. That is, eliminate any branch with upper bound smaller than $\frac{\hat{\mu}}{1-\epsilon}$ where $\hat{\mu}$ is the maximum record of the lower bounds of all branches ever generated and $\epsilon > 0$ is the tolerance. The final $\hat{\mu}$ is returned as the maximal structured singular value μ_{max}. Since $\hat{\mu}$ is the maximum record of the lower bounds obtained in all branches generated (no matter belong to the same frequency or not), it will increase much faster than its counterpart in the conventional frequency sweeping algorithms. Note that the raise of $\hat{\mu}$ results in a significant number of branches to be pruned. Thus $\hat{\mu}$ convergences quickly to the maximal structured singular value μ_{max}.

The paper is organized as follows. Section 2 discusses existing techniques for computing the maximal structured singular value. Section 3 presents our Parallel Branch and Bound Algorithm. An illustrative example is provided in Section 4 and Section 5 is the conclusion.

2 Conventional Frequency Sweeping

Consider an $M - \Delta$ set up as follows.

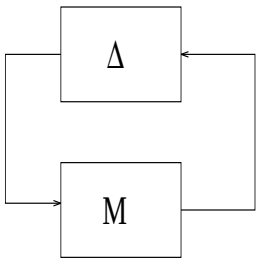

Figure 1: Uncertain System.

Let $\mathbf{\Delta}$ be a set of block structured uncertainties. We consider the computation of

$$\mu_{max} := \sup_{\omega \in \mathbf{R}} \mu_{\mathbf{\Delta}}(M(j\omega)).$$

For notation simplicity, let $M(\omega) = M(j\omega)$. Then $\mu_{max} = \sup_{\omega \in \mathbf{R}} \mu_{\mathbf{\Delta}}(M(\omega))$.

In practice, it is impossible to search μ_{max} over all frequencies. However, we can estimate μ_{max} as follows.

Choose a range of frequency $[a,\ b] \in \mathbf{R}$ and grid it as

$$\omega_j = a + \frac{(b-a)(j-1)}{NK-1}, \quad j = 1, \cdots, NK \tag{1}$$

where $N \geq 2$ and $K \geq 1$ are integers (In practice, gridding is usually based on the logarithmic scale. However, in this paper, we use uniform gridding for the simplicity of description.) Then an estimate for μ_{max} can be defined as

$$\tilde{\mu}_{max} := \max_{j=1,\cdots,NK} \mu_{\mathbf{\Delta}}(M(\omega_j)).$$

Define the (maximum positive real eigenvalue) function $\bar{\lambda}_R : \mathbf{C}^{n\times n} \to \mathbf{R}$ as

$$\bar{\lambda}_R(M) := \max\{\lambda : \ \lambda \text{ is a positive real eigenvalue of } M\}$$

with $\bar{\lambda}_R(M) = 0$ if M has no positive real eigenvalues. Let $\mathbf{B\Delta} := \{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) \leq 1\}$. Then

$$\mu_{\mathbf{\Delta}}(M) = \max_{\Delta\in\mathbf{B\Delta}} \bar{\lambda}_R(M\Delta).$$

Let $Q \subset \mathbf{B\Delta}$. Define μ on a box [9]

$$\mu(M, Q) := \max_{\Delta\in Q} \bar{\lambda}_R(M\Delta).$$

There exists techniques in [9] for computing an upper bound $UB(M, Q)$ and a lower bound $LB(M, Q)$ for $\mu(M, Q)$. Thus a branch and bound scheme can be applied to compute $\mu_{\mathbf{\Delta}}(M)$ with parameter space $\mathbf{B\Delta}$.

The conventional methods work essentially as follows. For $j = 1, \cdots, NK$, apply the following Algorithm 1 to compute an upper bound UB^j and a lower bound LB^j for $\mu_{\mathbf{\Delta}}(M(\omega_j))$ such that $UB^j - LB^j \leq \varepsilon$. Then $\tilde{\mu}_{max}$ satisfies

$$\max_{j=1,\cdots,NK} LB^j \leq \tilde{\mu}_{max} \leq \max_{j=1,\cdots,NK} UB^j$$

where

$$\max_{j=1,\cdots,NK} UB^j - \max_{j=1,\cdots,NK} LB^j \leq \varepsilon.$$

Algorithm 1 — **Branch and Bound ([9])**

Initialize Let $\mathcal{U}_j = \{Q_k\} = \mathbf{B\Delta}$.

Let

$$UB^j = \max_k UB(M(\omega_j), Q_k),$$
$$LB^j = \max_k LB(M(\omega_j), Q_k). \tag{2}$$

while $UB^j - LB^j > \varepsilon$

- Choose Q to be any element of $\mathcal{U}_j$ with $UB(M(\omega_j), Q) = UB^j$.
- Partition Q into Q_a and Q_b by bisecting along one of its longest edges.
- Add Q_a and Q_b into $\mathcal{U}_j$. Remove Q from $\mathcal{U}_j$.

- Remove from $\mathcal{U}_j$ any Q with

$$UB(M(\omega_j), Q) < LB^j. \tag{3}$$

endwhile

The most important mechanism of Algorithm 1 is "pruning" [9]. That is, any element of $\mathcal{U}_j$ for which (3) is satisfied will never again be partitioned and need not be considered further. We call inequality (3) as the "pruning condition".

We can see that existing techniques for computing $\tilde{\mu}_{max}$ employ branch and bound techniques for each frequency independently. In particular, the pruning process for one frequency is independent of another. $\mu_{\mathbf{\Delta}}(M)$ is bounded tightly for each frequency. Note that we usually need to evaluate $\mu_{\mathbf{\Delta}}(M)$ for many frequencies in order to obtain a reasonably good estimate of the maximal structured singular value μ_{max}. Thus the overall computation is still a heavy burden, even the computation of $\mu_{\mathbf{\Delta}}(M)$ for each frequency is very efficient.

Thus for the sake of efficiency, there is a strong motivation to conceive a smart frequency sweeping strategy. More specifically, we would raise the following question,

Is it possible to obtain the maximal structured singular value μ_{max} without tightly bounding $\mu_{\mathbf{\Delta}}(M(\omega_j))$ for each frequency ω_j?

The following section is devoted to answering this question.

3 Parallel Branch and Bound Algorithm

It is fair to compare the performance of different algorithms on the same set of frequencies. Therefore, we consider again frequencies $\omega_j,\ \ j = 1, \cdots, NK$ defined by (1) and relabel them as

$$\omega_{ij} := a + \frac{(b-a)[K(i-1) + (j-1)]}{NK - 1}, \quad i = 1, \cdots, N,\ \ j = 1, \cdots, K.$$

Now we are in a good position to present our Parallel Branch and Bound Algorithm as follows.

Algorithm 2 — **Parallel Branch and Bound Algorithm**

- Step 1: Initialize. Set $j = 1$. Set $\hat{\mu} = 0$. Set tolerance $\epsilon > 0$. Set maximal iteration number IT.

- Step 2: Update $\hat{\mu}$ and record the number of iterations $r(j)$ for frequency ω_{ij} by the following steps.

  - Step 2–1: Let $\mathcal{U}_{ij} = \{Q_k\} = \mathbf{B\Delta},\ \ i = 1, \cdots, N$. Set $r = 1$.
  - Step 2–2: If $r = IT + 1$ or $\mathcal{U}_{ij}$ is empty for any $i \in \{1, \cdots, N\}$ then record $r(j) = r$ and go to Step 3, else do the following for all i such that $\mathcal{U}_{ij}$ is not empty.

∗ Choose Q to be any element of $\mathcal{U}_{ij}$ with

$$UB(M(\omega_{ij}), Q) = \max_{Q_k \in \mathcal{U}_{ij}} UB(M(\omega_{ij}), Q_k).$$

∗ Partition Q into Q_a and Q_b by bisecting along one of its longest edges.

∗ Add Q_a and Q_b into $\mathcal{U}_{ij}$. Remove Q from $\mathcal{U}_{ij}$.

∗ Update

$$\hat{\mu} = \max\{\hat{\mu},\ LB(M(\omega_{ij}), Q_a),\ LB(M(\omega_{ij}), Q_b)\}. \tag{4}$$

∗ Remove from $\mathcal{U}_{ij}$ any Q with

$$UB(M(\omega_{ij}), Q) < \frac{\hat{\mu}}{1-\epsilon}. \tag{5}$$

– Step 2–3: Set $r = r + 1$ and go to Step 2–2.

• Step 3: If $j = K$ then STOP, else set $j = j + 1$ and go to Step 2.

In Algorithm 2, N branches of frequency sweeping are performed in parallel with starting frequencies $\omega_{i1}, \quad i = 1, \cdots, N$ and step size $\frac{b-a}{NK-1}$. Also, a branch and bound scheme is applied to compute μ for N frequencies in parallel. Any branch with upper bound smaller than $\frac{\hat{\mu}}{1-\epsilon}$ will be pruned, where $\hat{\mu}$ is the maximum record of the lower bounds of all branches ever generated. The final $\hat{\mu}$ is returned as the maximal structured singular value μ_{max}. Algorithm 2 is visulized in the following Figure 2.

Remark 1 *Note that Algorithm* 2 *provides a substantial improvement on efficiency than conventional methods in computing the maximal structured singular value. This can be explained by the significant relaxation in the "pruning condition" of Algorithm* 2*. To see the difference of the two "pruning conditions", we can compare the right hand sides of inequalities (5) and (3). By (4) and (2), we can see that* $\frac{\hat{\mu}}{1-\epsilon}$ *can be much larger than* LB^j*. This is because* $\hat{\mu}$ *is the maximum record of the lower bounds obtained in all branches of all frequencies evaluated and being evaluated, while* LB^j *is only the maximum record of the lower bounds obtained in branches of the frequency being evaluated. Moreover,* $\hat{\mu}$ *is enlarged to* $\frac{\hat{\mu}}{1-\epsilon}$ *in the "pruning condition" (5) and hence the "pruning" process is further facilitated. The significant relaxation of the 'pruning condition" leads to a substantial decrease of the number of total subdomains needed to be evaluated. Therefore, our algorithm is much more efficient than those previously available to control engineers.*

Remark 2 *It is important to note that Algorithm* 2 *involves only one CPU processor. It is fundamentally different from the parallel algorithms which involves more than one CPU processors.*

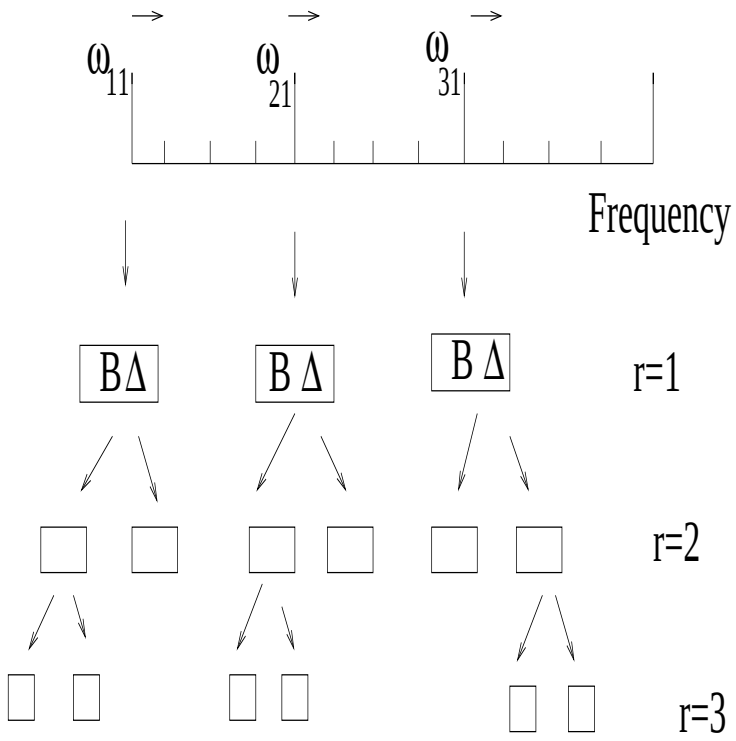

Figure 2: A Picture of Parallel Branch and Bound Algorithm. $N = 3,\ \ K = 4$.

Remark 3 *A substantial amount of computation can be saved by the following mechanisms. First, further computation of the lower bound on a domain is not needed once it is determined that the lower bound is smaller than the existing global lower bound. This can be seen from equation (4). Second, the computation of the upper bound should be terminated once condition (5) is satisfied. The idea of these two mechanisms is to avoid as much as possible the tightly computation of the lower bound and the upper bound.*

In addition to the novel frequency sweeping strategy, another character of Algorithm 2 is that there is no tolerance criteria directly forced on the final result, however, the final result falls into tolerance automatically.

Theorem 1 *Suppose that the maximal iteration number $IT < \infty$ and that Algorithm 2 stops with $r(j) \leq IT,\ \ j = 1, \cdots, K$. Then the final $\hat{\mu}$ satisfies*

$$0 \leq \frac{\tilde{\mu}_{max} - \hat{\mu}}{\tilde{\mu}_{max}} < \epsilon.$$

Proof. Since $\hat{\mu}$ is the maximal record of the lower bounds, we have $\tilde{\mu}_{max} - \hat{\mu} \geq 0$. We only need to show that $\frac{\tilde{\mu}_{max} - \hat{\mu}}{\tilde{\mu}_{max}} < \epsilon$. By the assumption that Algorithm 2 stops with $r(j) \leq IT,\ \ j = 1, \cdots, K$, we know that all subdomains ever generated are finally removed because the "pruning condition" (5) is

satisfied. Note that there exists a subdomains Q_{ij} for frequency ω_{ij} such that $\mu(M(\omega_{ij}), Q_{ij}) = \tilde{\mu}_{max}$. Let $\hat{\mu} = \bar{\mu}$ when Q_{ij} is removed. Then $\tilde{\mu}_{max} \leq UB(M(\omega_{ij}), Q_{ij}) < \frac{\bar{\mu}}{1-\epsilon}$. Note that $\hat{\mu}$ is nondecreasing thus the final $\hat{\mu} \geq \bar{\mu}$. It follows that

$$\tilde{\mu}_{max} < \frac{\hat{\mu}}{1-\epsilon} \quad \Longrightarrow \quad \frac{\tilde{\mu}_{max} - \hat{\mu}}{\tilde{\mu}_{max}} < \epsilon.$$

The proof is thus completed.

□

Note that one important concern of an algorithm is convergence. It is usually desirable that, given any tolerance $\epsilon > 0$, an algorithm stops and returns the result within tolerance in a finite number of iterations. Obviously, the convergence requirement imposes condition of the quality of bounds.

Definition 1 *The upper bound $UB(M,.)$ and lower bound $LB(M,.)$ are said to be continuous if*

$$\lim_{d(Q)\to 0} UB(M,Q) - LB(M,Q) = 0$$

where $d(Q) := \max_{q,\, q' \in Q} ||q - q'||$ with $Q \subseteq \mathbf{B\Delta}$.

Theorem 2 *Suppose that all the upper bounds and lower bounds are continuous and that at least one nonzero lower bound appears after a finite number of iterations. Let the maximal iteration number $IT = \infty$. Then, for arbitrary tolerance $\epsilon > 0$, Algorithm 2 stops with a finite number of domain partitions for each j, i.e., $r(j) < \infty$, $j = 1, \cdots, K$. Moreover, the final $\hat{\mu}$ satisfies*

$$0 \leq \frac{\tilde{\mu}_{max} - \hat{\mu}}{\tilde{\mu}_{max}} < \epsilon.$$

Proof. Suppose that Algorithm 2 does not stop with a finite number of domain partitions for each j. Then $\exists\ \omega_{ij}$ and an infinite sequence of nested subdomains $\{Q_r^{ij}\}$ associated with frequency ω_{ij} such that $Q_1^{ij} \supset Q_2^{ij} \supset \cdots \supset Q_r^{ij} \supset \cdots$. Note that by the assumption $\exists r_0 < \infty, \mu_0 > 0$ such that $\hat{\mu} \geq \mu_0$, $\forall r > r_0$. By the continuity, $\exists r_1$ such that

$$UB(M(\omega_{ij}), Q_r^{ij}) - LB(M(\omega_{ij}), Q_r^{ij}) < \frac{\epsilon}{1-\epsilon}\mu_0, \quad \forall r > r_1.$$

Let $r_2 = \max\{r_0, r_1\} + 1$. Then

$$UB(M(\omega_{ij}), Q_{r_2}^{ij}) - LB(M(\omega_{ij}), Q_{r_2}^{ij}) < \frac{\epsilon}{1-\epsilon}\mu_0.$$

Thus

$$UB(M(\omega_{ij}), Q_{r_2+1}^{ij}) - \hat{\mu} < \frac{\epsilon}{1-\epsilon}\hat{\mu} \quad \Longrightarrow \quad UB(M(\omega_{ij}), Q_{r_2+1}^{ij}) < \frac{\hat{\mu}}{1-\epsilon}$$

which implies that $Q^{ij}_{r_2+1}$ is removed. This is a contradiction. Therefore Algorithm 2 stops with a finite number of domain partitions for each j and hence by the same argument of Theorem 1

$$0 \leq \frac{\tilde{\mu}_{max} - \hat{\mu}}{\tilde{\mu}_{max}} < \epsilon.$$

The proof is thus completed. □

4 An Illustrative Example

Consider an $M - \Delta$ set up as shown in Figure 1 where $\Delta = \text{diag}(\delta_1, \delta_2) \in \mathbf{R}^{2\times 2}$ and $M(s) = C(sI - A)^{-1}B$ with

$$A = \begin{bmatrix} -1 & -10 & -1 & 10 \\ -0.5 & -1 & 1 & 0.5 \\ 0.5 & -4 & -1 & -10 \\ -10 & 0.5 & 0 & -2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.5 \end{bmatrix}.$$

To compute the supremum of μ, we uniformly grid frequency interval $[0.01,\ 15.01]$ and obtain $1,500$ grid frequencies as

$$\omega_j = 0.01 + \frac{(15.01 - 0.01)(j-1)}{1500 - 1}, \quad j = 1, \cdots, 1500.$$

In Algorithm 2, we choose the relative error $\epsilon = 0.01$ and $N = 30,\ \ K = 50$. The $1,500$ frequencies are regrouped as

$$\omega_{ij} = 0.01 + \frac{(15.01 - 0.01)[50(i-1) + (j-1)]}{1500 - 1}, \quad i = 1, \cdots, 30; \quad j = 1, \cdots, 50.$$

The execution of Algorithm 2 is terminated at $r = 1$ with $\hat{\mu} = 0.8424$ achieved at frequency $\omega_{ij} = 9.1661$ where $i = 19,\ j = 16$. It is observed that, for any frequency, no partition is performed for the original domain $\mathbf{B\Delta} = [-1, 1] \times [-1, 1]$. This is because $\frac{\hat{\mu}}{1-\epsilon}$ is greater than the upper bounds of singular values for $\mathbf{B\Delta}$ at other frequencies. The bounds of singular values for $\mathbf{B\Delta}$ are shown by Figures 3-4. It can be seen from these figures that the upper bounds and lower bounds of singular values are far apart for most of the frequencies. To compute the maximal singular value using the conventional branch and bound method, substantial computational effort will be wasted on reducing the gap between the upper bounds and lower bounds of singular values for most of the frequencies. This example demonstrates that branch and bound should not be applied extensively for any fixed frequency. Quiet contrary, it should be employed in parallel and in a cooperative manner. This spirit has been reflected in Algorithm 2.

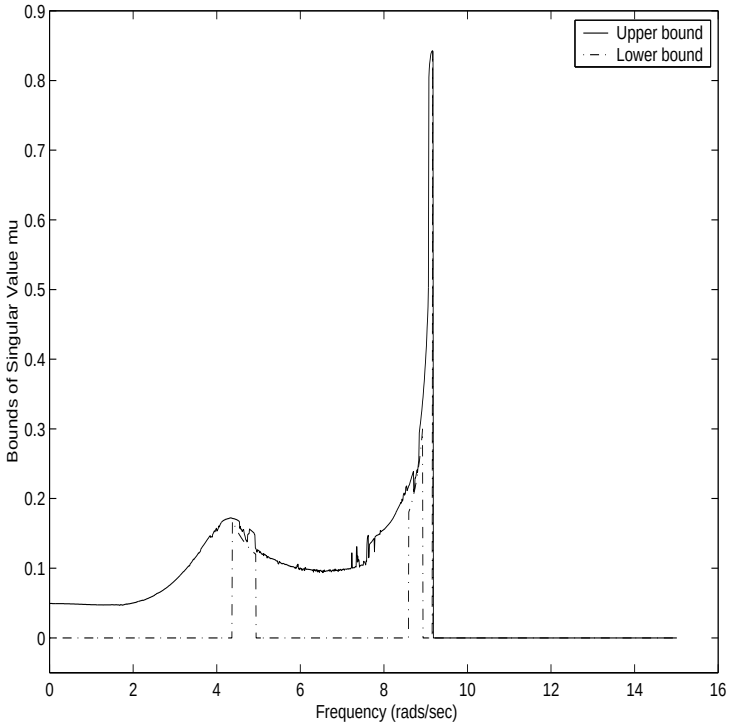

Figure 3: Bounds of Singular Values

5 Conclusion

Efficient computation of the maximal structured singular value is of fundamental importance in robustness analysis and robust synthesis with structured uncertainty. Motivated by this, we have developed a parallel branch and bound algorithm for computing the maximal structured singular value, which significantly reduce the computational complexity.

References

[1] Balas, G., Doyle, J. C., Glover, K., Packard, A. and Smith, R., *μ-Analysis and Synthesis Toolbox*, MUSYN Inc. and The MathWorks, Inc. 1995.

[2] Balas, G. and Packard A., "The structured singular value (μ) framework," *The Control Hand Book*, pp. 671-687, CRC Press, Inc, 1996.

[3] Doyle, J. C., "Analysis of feedback system with structured uncertainty," *IEE Proc.*, pt. D, vol. 129, no. 6, pp. 242-250, 1982.

[4] Fan, M. K. H. and Tits, A. L., "Characterization and efficient computation of the structured singular value," *IEEE Trans. Autom. Control*, vol. 31, pp. 734-743, 1986.

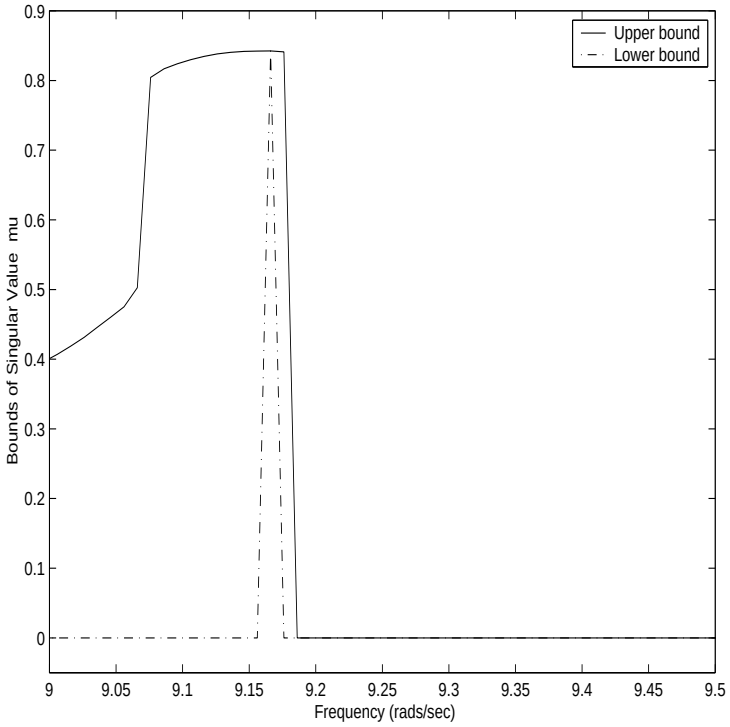

Figure 4: Bounds of Singular Values

[5] Ferrers and Biannic, "A μ-analysis technique without frequency gridding," *Proceeding of American Control Conference*, pp. 2294-2298, 1998.

[6] Ferrers and Biannic, "Reliable computation of the robustness margin for a flexible aircraft," *Control Engineering Practice*, pp. 1267-1278, December 2001.

[7] A. Helmersson, "A finite frequency method for μ-analysis," *Proceeding of European Control Conference*, pp. 171-176, 1995.

[8] Lawrence, C. T., Tits, A. L. and Dooren, P. V., "A fast algorithm for the computation of an upper bound on the μ-norm," *Automatica*, vol. 36, pp. 449-456, 2000.

[9] Newlin M. P. and Young P. M., "Mixed μ problems and branch and bound techniques," *Proc. IEEE Conference on Decision and Control ,* pp. 3175-3180, Tucson, Arizona, 1992.

[10] Packard, A. and Doyle, J. C., "The complex structured singular value," *Automatica*, 29(1), pp. 71-109, 1993.

[11] Zhou, K., Doyle, J. C. and Glover, K., *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ, 1996.